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A GAS PENDULUM

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It is shown that a periodic two-dimensional isentropic motion of a gas exists and it is described by an exact solution of the equations of gas dynamics. A polytropic gas that fills a circular cylinder rotates and oscillates (in the radial direction) simultaneously under the action of periodically changing external pressure. The solution obtained belongs to the class of solutions with a velocity field that is linear in the coordinates (with homogeneous deformation).

Introduction. The regimes of motion of a gas (or a liquid) under homogeneous deformation has long been studied and is dealt with in a large body of literature (see, e.g., [1]). We recall that for a polytropic gas with an adiabatic exponent $\gamma > 1$, one class of these regimes of motion is governed by the formulas

$$x = M\xi, \qquad \boldsymbol{u} = \dot{M}\xi, \qquad p = (\gamma - 1)m^{-\gamma}g(h), \qquad \rho = m^{-1}g'(h), \tag{0.1}$$

where $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^n$ are, respectively, the Cartesian and the Lagrangian coordinates of the gas particles, $\boldsymbol{u}$ is the velocity vector, $M = M(t)$ is an $(n \times n)$ nondegenerate matrix ($n = 2, 3$), $g(h) > 0$ is an arbitrary differentiable function, $g'(h) > 0$ is its derivative; the quantities $m(t)$ and $h(\xi)$ are such that:

$$m = \det M > 0, \qquad h = \varepsilon|\xi|^2/2 \qquad (\varepsilon = \pm 1). \tag{0.2}$$

The dot above M and other quantities denotes differentiation with respect to time t. The quantities (0.1) and (0.2) are an exact solution of the equations of gas dynamics provided that the matrix M satisfies the equation

$$\ddot{M} + \varepsilon(\gamma - 1)m^{1-\gamma}M_*^{-1} = 0, \tag{0.3}$$

in which M_*^{-1} is the inverse of the transposed matrix M_*.

Equation (0.3) is a dynamic system of order $2n^2$ for the n^2 elements of the matrix $M = (M_{ij})$. This system has $n^2 - n + 1$ first integrals, which were first found in [2], namely, the integrals of the moment of momentum

$$M\dot{M}_* - \dot{M}M_* = J,$$

the vorticity integrals

$$M_*\dot{M} - \dot{M}_*M = \mathcal{K}$$

and the energy integral

$$\frac{1}{2}\sum_{i,j}\dot{M}_{ij}^2 = \varepsilon m^{1-\gamma} + E,$$

where J and $\mathcal{K}$ are $(n \times n)$ arbitrary constant skew-symmetric matrices and E is an arbitrary constant (for a more general case, the integrals $\mathcal{K}$ and E were obtained in [3]). Anisimov and Lysikov [4] inferred that for $\gamma = (n + 2)/n$, system (0.3) has the additional integrals

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$$\sum_{i,j} M_{ij}^2 = 2Et^2 + At + B,$$

where A and B are constants. In the case $n = 2$ ($\gamma = 2$), these were used in [4] to show that the corresponding system (0.3) can be integrated in quadratures.

The quantity $\varepsilon = \pm 1$ and the function $g(h)$ can be used to develop various specific physical models for motion of a gas. For example, for $\varepsilon = -1$, Dyson [2] considered a "gas cloud" model that corresponds to $g = \exp(h)$. Moreover, for $\varepsilon = -1$, a choice of a finite function $g(h)$ gives examples of expansion of a gas ellipsoid into vacuum. Setting $\varepsilon = 1$, one can consider the motion of a gas ellipsoid under the action of external pressure that varies with time. Isentropic motion of a gas corresponds to the choice of g according to the relation $g(g')^{-\gamma} = \text{const}$, which leads to the function

$$g = (c_0 + c_1 h)^{\gamma/(\gamma-1)} \qquad (c_0, c_1 = \text{const}). \tag{0.4}$$

Below, we consider a simple example of an exact solution that describes the two-dimensional motion ($n = 2$) of a gas cylinder under the action of external pressure.

1. Model of Motion of a Cylinder. For $n = 2$, we consider the matrix

$$M = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \qquad M_*^{-1} = m^{-1}M, \qquad m = a^2 + b^2. \tag{1.1}$$

In this case, the dynamic system (0.3) reduces to the equations

$$\ddot{a} + \varepsilon(\gamma - 1)m^{-\gamma}a = 0, \qquad \ddot{b} + \varepsilon(\gamma - 1)m^{-\gamma}b = 0. \tag{1.2}$$

Here J and $\mathcal{K}$ satisfy the relation $J + \mathcal{K} = 0$, which implies that system (1.2) has the two first integrals

$$b\dot{a} - a\dot{b} = j \quad (j = \text{const}), \qquad \dot{a}^2 + \dot{b}^2 = \varepsilon m^{1-\gamma} + 2E. \tag{1.3}$$

In the polar coordinates

$$a = m^{1/2}\cos\beta, \qquad b = m^{1/2}\sin\beta, \tag{1.4}$$

system (1.3) becomes

$$\dot{\beta} = -j/m; \tag{1.5}$$

$$\dot{m}^2 = 4f(m), \qquad f(m) = \varepsilon m^{2-\gamma} + 2Em - j^2. \tag{1.6}$$

Thus, the dynamic system (1.2) can be integrated in quadratures.

The kinematics of motion of the gas described by these solutions can be clearly demonstrated with the use of the polar coordinates

$$(x, y) = r(\cos\varphi, \sin\varphi), \qquad (\xi, \eta) = \sigma(\cos\theta, \sin\theta)$$

in terms of the radial (V_r) and the circumferential (V_φ) components of the velocity vector $\boldsymbol{u} = (u, v)$:

$$V_r = u\cos\varphi + v\sin\varphi, \qquad V_\varphi = -u\sin\varphi + v\cos\varphi.$$

Using Eqs. (1.3)–(1.5), one readily obtains the expressions

$$V_r = r\dot{m}/(2m), \qquad V_\varphi = rj/m, \qquad r = \sigma m^{1/2}. \tag{1.7}$$

It follows that at each moment, the distribution of the circumferential velocities is identical to that for rigid-body rotation with angular velocity $\Omega = j/m$ and the distribution of the radial velocities is proportional to the radius r. Without loss of generality, we assume that the region occupied by the gas is bounded by the surface of a cylinder C_R of radius R that corresponds to the value $\sigma = 1$, i.e., $R = r\big|_{\sigma=1}$. Then, $R = [m(t)]^{1/2}$, the radial velocity of motion of the surface C_R is

$$V_R = R\dot{m}/(2m), \tag{1.8}$$

and the entire cylinder rotates with angular velocity $\Omega = j/m$. In this case, the region occupied by the gas is assumed to be inside C_R, i.e., for $r < R$ or $\sigma < 1$. Using formula (0.1), we determine the pressure on C_R:

$$p_R = (\gamma - 1)m^{-\gamma} g(1/2). \tag{1.9}$$

Determination of the possible modes of specific motion of the gas in model (1.1) is reduced to analysis of solutions of the key equation (1.6). They depend strongly on the signs of the three quantities: ε, $\gamma - 2$, and E.

Real solutions of Eq. (1.6) are possible only in the intervals $\Delta \in (0 < m < \infty)$ in which $f(m) > 0$. For $\varepsilon = -1$, the condition $E > 0$ must hold. In this case, for any $\gamma > 1$, an interval $\Delta = (m_0, \infty)$ with $m_0 > 0$ exists, and the corresponding solutions describe unbounded expansion of the gas cylinder C_R into vacuum. For $\varepsilon = 1$, $\gamma \geqslant 2$, and any E, there exists an interval $\Delta = (0, m_0)$ which corresponds to the motion of C_R with collapse of the density and pressure on the axis $r = 0$. If $\varepsilon = 1$ and $\gamma < 2$ an oscillation mode of the motion of C_R is possible, which is discussed in Sec. 2.

2. Pulsation of the Cylinder. Let $\varepsilon = 1$, $\gamma < 2$, and $E < 0$. In this case, the expressions

$$f'(m) = (2-\gamma)m^{1-\gamma} + 2E, \qquad f''(m) = (2-\gamma)(1-\gamma)m^{-\gamma}$$

imply that the function $f(m)$ is convex upward and has a maximum at the point m_* determined from the relation

$$(2-\gamma)m_*^{1-\gamma} = 2|E|. \tag{2.1}$$

At this point, the function f has the value

$$f(m_*) = \frac{\gamma - 1}{2 - \gamma} 2|E|m_* - j^2. \tag{2.2}$$

Therefore, if quantity (2.2) is positive, there exists an interval $\Delta = (m_1, m_2)$, where $0 < m_1 < m < m_2 < \infty$, in which $f(m) > 0$ and $f(m_1) = f(m_2) = 0$. Moreover, $f'(m_1) > 0$ and $f'(m_2) < 0$. Consequently, the representation

$$f(m) = (m - m_1)(m_2 - m)F(m) \tag{2.3}$$

is valid. Here $F(m) > 0$ for $m_1 \leqslant m \leqslant m_2$, and the quadrature

$$\pm \int \frac{dm}{\sqrt{f(m)}} = 2t + C \tag{2.4}$$

determines a periodic function $m(t)$ with period

$$T = \int_{m_1}^{m_2} \frac{dm}{\sqrt{f(m)}}. \tag{2.5}$$

Let $m(0) = m_1$ for $t = 0$, which corresponds to the minimum radius $R = \sqrt{m_1}$ of the cylinder C_R and the maximum external pressure on it determined by (1.9). For $t = 0$, the gas rotates as a rigid body with maximum angular velocity $\Omega_1 = j/m_1$. For $t > 0$, expansion of the cylinder begins according to the law $R = \sqrt{m(t)}$, where $m(t)$ is determined by the following formula [upper sign (plus) in (2.4)]:

$$\int_{m_1}^{m(t)} \frac{dm}{\sqrt{f(m)}} = 2t,$$

and the pressure on the cylinder wall decreases monotonically until the value $m = m_2$ is reached at the moment $t_1 = T/2$, when C_R rotates again as a rigid body with angular velocity $\Omega_2 = j/m_2 < \Omega_1$, and the pressure on C_R is minimum. After that, compression of the cylinder occurs: $m(t)$ decreases according to the formula [lower sign (minus) in (2.4)]

$$\int\limits_{m(t)}^{m_2} \frac{dm}{\sqrt{f(m)}} = 2(t - t_1).$$

The compression occurs up to the moment t_2, when $m(t)$ reaches the value m_1 once again; in this case, t_2 is determined from the relation $2t_1 = 2(t_2 - t_1)$, whence $t_2 = T$. For $t = t_2$, the motion assumes the initial mode (as for $t = 0$), and further, the process is repeated with period T. This motion can be called a "gas pendulum" by analogy with the usual mechanical oscillations of a physical pendulum. The principal specific feature of the "gas pendulum" is that it exists "eternally," similarly to steady gas flows: it cannot be obtained from the rigid-body rotation of the gas by external action on the boundary C_R over a finite time. Indeed, the variation of the parameters of motion due to this action must propagate into the cylinder *with a finite velocity*, whereas the motion mode of the "gas pendulum" is determined immediately in the entire cylinder C_R. In view of this, experimental implementation of the "gas pendulum" can involve certain difficulties.

The pressure distribution in the "gas pendulum" is determined up to an arbitrary function $g(h)$. In particular, an *isentropic* "gas pendulum" is possible, where the pressure is given by formulas (0.1) and (0.4). Moreover, a *vacuum region* ($r = 0$) can occur in the neighborhood of the axis ($p = 0$); to obtain this region, it suffices to choose values $c_0 < 0$ and $c_1 > 2|c_0|$ in (0.4).

3. Normalization and Example. The dynamic system (0.3) admits (in the Lie sense) a large group of transformations of the space $\mathbb{R}^{10}(t, M)$, which deserves a separate discussion. Here we use only an admissible one-parameter group of extension, which, in the case of two-dimensional motion ($n = 2$), is given by the formulas

$$t = k^{\gamma} t', \qquad M = kM', \qquad m = k^2 m' \tag{3.1}$$

with the extension parameter $k > 0$. As applied to system (1.1)–(1.3), this leads to the following transformation of the constants:

$$j = k^{2-\gamma} j', \qquad E = k^{2-2\gamma} E'. \tag{3.2}$$

One can easily verify that the key equation (1.6) and its integrals (1.3) remain invariant for transformations (3.1) and (3.2).

Using (3.1), one can normalize the solution so that $m'_* = 1$, for which purpose it is necessary to set $k = \sqrt{m_*}$. According to (2.1), the normalized solution is determined by Eq. (1.6) with $2|E| = 2 - \gamma$, i.e., with the function

$$f(m) = m^{2-\gamma} - (2 - \gamma)m - j^2, \tag{3.3}$$

and according to (2.2), the condition of positive quantity $f(m_*) = f(1)$ assumes the form $\gamma > 1 + j^2$. It follows that a normalized "gas pendulum" exists if $j^2 < \gamma - 1 < 1$ and oscillates with period T (2.5), where $f(m)$ is given by formula (3.3).

As an example, we consider the value $\gamma = 3/2$, for which the period T is calculated exactly and is independent of the quantity $j^2 < 1/2$. Indeed, if one introduces the radius $R = \sqrt{m}$ of the cylinder C_R in place of m, for $\gamma = 3/2$, Eq. (1.6) becomes

$$2R^2\dot{R}^2 = (R - R_1)(R_2 - R),$$

where

$$R_1 = 1 - \sqrt{1 - 2j^2}, \qquad R_2 = 1 + \sqrt{1 - 2j^2}. \tag{3.4}$$

The quadrature yields the relation between t and R

$$2 \arcsin \sqrt{\frac{R - R_1}{R_2 - R_1}} - \sqrt{(R - R_1)(R_2 - R)} = t/\sqrt{2} \qquad (R_1 \leqslant R \leqslant R_2) \tag{3.5}$$

and the period $T = 2\sqrt{2}\pi$.

The natural question of the existence of a spherical ($n = 3$) "gas pendulum" is still an open question.

REFERENCES

1. O. I. Bogoyavlenskii, "Methods of qualitative theory of dynamic systems in astrophysics and gas dynamics," in: *Dynamics of Gas Ellipsoids* [in Russian], Nauka, Moscow (1980), pp. 261–299.
2. F. J. Dyson, "Dynamics of a spinning gas cloud," *J. Math. Mech.*, **18**, No. 1, 91–101 (1968).
3. L. V. Ovsyannikov, "New solution of hydrodynamic equations," *Dokl. Akad. Nauk SSSR*, **111**, No. 1, 47–49 (1956).
4. S. I. Anisimov and Yu. I. Lysikov, "Expansion of a gas cloud into vacuum," *Prikl. Mat. Mekh.*, **34**, 926–929 (1970).